Cascade Directional Filter*

OMAR WING†

Summary—A directional filter is a completely matched four-port which exhibits a directional and a filter-like frequency characteristic. This paper explores the properties of N-directional filters connected in cascade through sections of transmission lines. Analysis shows that if a directional filter admits the equivalent circuit representation offered here, its transfer functions are functions of only one parameter, a susceptance function. When the directional filters are cascaded in a certain way, the over-all transfer functions have the same form as before except that the susceptance function is now the sum of the susceptance functions of the component filters. The last property is an important one. Given a transfer function expressed in terms of a susceptance function, the network designer can expand the susceptance in partial fraction and realize the transfer function using directional filters in cascade, each being characterized by a much simpler susceptance.

Introduction

A directional filter is a completely matched four-port which exhibits a directional and a filter-like frequency characteristic. It may take any of several different physical forms.[1–4] In all cases, it has the following well-known properties:

1) It is reflectionless, *i.e.*, all four-ports are matched when they are terminated in their own characteristic impedances.
2) It is directional, *i.e.*, signal entering into port 1 emerges at ports 2 and 3, none at 4, etc.
3) The transfer function between ports 1 and 2 and that between 1 and 3 are complements of each other, *i.e.*, if one has a band-pass characteristic, the other has a band-elimination characteristic.

This paper explores the properties of N directional filters connected in cascade through sections of transmission line. Analysis shows that if a directional filter admits the equivalent circuit representation offered here, its transfer functions are functions of one parameter, a susceptance function. When the directional filters are cascaded in a certain way, the over-all transfer functions have the same form as before except that the susceptance function is now the sum of the susceptance functions of the component directional filters. The last property is an important one, for it offers the network designer flexibility and convenience in realizing a transfer function. Given a transfer function in a form suitable for realization in a directional filter, he can synthesize the network using several directional filters in cascade, each being characterized by a much simpler susceptance function.

Transfer Functions of a Directional Filter

Let a directional filter admit an equivalent representation shown in Fig. 1. It consists of four two-port

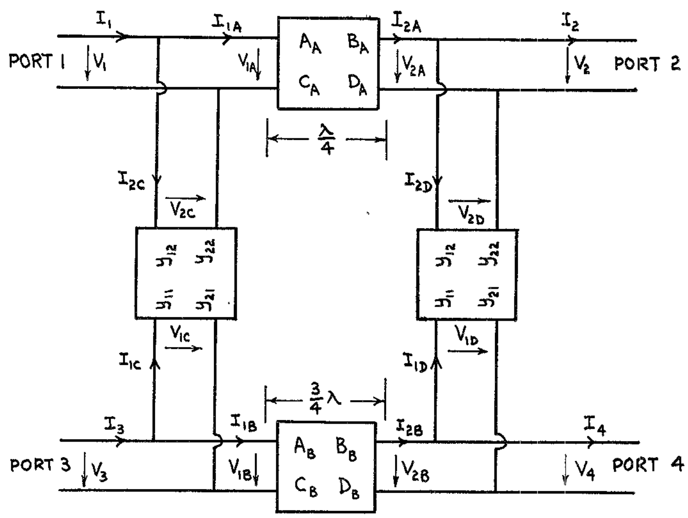

Fig. 1—Schematic diagram of a directional filter.

networks inter-connected to form a network having two input terminals and two output terminals. For a reason which will be apparent later, the two series two-ports are characterized by their $ABCD$ matrix and the two shunt two-ports are characterized by their short-circuit admittance matrix. Since we are interested in connecting N of these networks in cascade, the obvious characterization of the filter is the generalized $ABCD$ matrix:

$$\begin{bmatrix} V_1 \\ V_3 \\ I_1 \\ I_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ V_4 \\ I_2 \\ I_4 \end{bmatrix}. \quad (1)$$

In practice, the two shunt two-ports are identical and the two series two-ports are sections of transmission lines $\lambda/4$ and $3\lambda/4$ long. As shown in Appendix I, under these conditions the generalized $ABCD$ matrix takes the following form:

* Manuscript received by the PGMTT, June 2, 1958; revised manuscript received, August 25, 1958. This paper was presented orally at the joint URSI-IRE conference held in Washington, D. C., on April 24, 1958. Some of the results reported here were obtained while the author was employed at the Bell Telephone Labs., Inc., New York 14, N. Y., in the summer of 1957.

† Dept. of Elec. Eng., Columbia University, New York, N. Y.

[1] S. B. Cohn and F. S. Coale, "Directional channel separation filters," Proc. IRE, vol. 44, p. 1018–1024; August, 1956.

[2] R. W. Klopfenstein and J. Epstein, "The polarguide—a constant resistance waveguide filter," Proc. IRE, vol. 44, pp. 210–218; February, 1956.

[3] F. S. Coale, "A traveling-wave directional filter," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 256–260; October, 1956.

[4] C. E. Nelson, "Circularly polarized microwave cavity filters," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 136–147; April, 1957.

$$M = \begin{bmatrix} jZ_Ay_{22} & jZ_Ay_{12} & -jZ_A & 0 \\ -jZ_By_{12} & -jZ_By_{11} & 0 & jZ_B \\ -j\dfrac{1}{Z_A} - jZ_Ay_{22}^2 + jZ_By_{12}^2 & jZ_By_{11}y_{12} - jZ_Ay_{12}y_{22} & \multicolumn{2}{c}{jZ_Ay_{22} - jZ_By_{12}} \\ -jZ_Ay_{22}y_{12} + jZ_By_{12}y_{11} & \dfrac{1}{Z_B} - jZ_Ay_{21}^2 + jZ_By_{11}^2 & \multicolumn{2}{c}{jZ_Ay_{12} - jZ_By_{11}} \end{bmatrix} \tag{2}$$

where

Z_A = characteristic impedance of transmission line 1.
Z_B = characteristic impedance of transmission line 2.
y_{11}, y_{12}, y_{22} = short-circuit admittance parameters of the two-port.

In operation, the ports are terminated in their own characteristic impedances, so that the terminal voltages and current satisfy a set of constraints given by

$$\begin{aligned} V_1 &= E - I_1Z_A \\ V_2 &= Z_AI_2 \\ V_3 &= -Z_BI_3 \\ V_4 &= Z_BI_4. \end{aligned} \tag{3}$$

We shall now derive the four transfer voltage ratios and from these deduce the necessary and sufficient conditions that the parameters of y_{11}, y_{12}, and y_{22} must satisfy in order to realize the directional and reflectionless properties mentioned earlier. As shown in Appendix II, we have

$$\frac{V_4}{E} = \frac{jZ_By_{12}(Z_By_{11} - Z_Ay_{22})}{\Delta} \tag{4}$$

where

$$\Delta = (2 + Z_Ay_{22} + Z_By_{11} + Z_AZ_By_{11}y_{22} - Z_AZ_By_{12}^2)^2. \tag{5}$$

It is clear from (4) that the directional property is realized if

$$Z_By_{11} = Z_Ay_{22}. \tag{6}$$

Using (6) we find

$$\frac{V_1}{E} = \frac{2+4Z_Ay_{22}+2Z_A^2y_{22}^2 + Z_A^2y_{22}^2(Z_Ay_{22} - Z_Ay_{12}^2/y_{11})}{\Delta}.$$

The reflectionless property is realized if

$$y_{11}y_{22} = y_{12}^2 \tag{7}$$

for then $V_1/E = \frac{1}{2}$, and port 1 is matched. In fact, by a well-known theorem[5] on the properties of a directional coupler, all four ports are matched.

Eq. (7) restricts the structure of the two-terminal network to be one whose usual open circuit impedance parameters do not exist. The simplest realization of the network is one which has only series admittance as shown in Fig. 2.

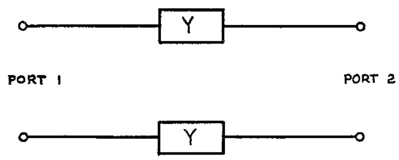

Fig. 2—The simplest realization of the shunt two-port.

Using (6) and (7), we find the other two transfer voltage ratios to be

$$\frac{V_2}{E} = -\frac{j}{2}\,\frac{1}{1 + Z_Ay_{22}} = -\frac{j}{2}\,\frac{1}{1 + Z_By_{11}}, \tag{8}$$

$$\begin{aligned} \frac{V_3}{E} &= -\frac{1}{2}\sqrt{\frac{Z_B}{Z_A}}\,\frac{Z_Ay_{22}}{1 + Z_Ay_{22}} \\ &= -\frac{1}{2}\sqrt{\frac{Z_B}{Z_A}}\,\frac{Z_By_{11}}{1 + Z_By_{11}}. \end{aligned} \tag{9}$$

It is seen that the two voltages are 90° out of phase—a well-known property of directional couplers. Moreover, let P_2 be the power delivered to the load Z_A at port 2 and P_3 be that to Z_B at port 3. Then using (8) and (9), we have

$$P_2 + P_3 = \frac{E^2}{4Z_A} = \text{constant}.$$

The two transfer characteristics are therefore complements of each other. In particular, if y_{22} is a simple resonant circuit, V_2/E exhibits a band elimination characteristic while V_3/E one of the bandpass.

The above analysis shows that a directional filter is completely characterized by one parameter, the susceptance function y_{22} (or alternately, y_{11}), as asserted earlier. Except for the factor $-j$, in all aspects, V_2/E as given by (8) is much like the transfer function of a constant-resistance bridge-T network.[6]

Cascade Directional Filter

If the two-port network parameters satisfy (6) and (7), the $ABCD$ matrix, M, is simplified to the following:

[5] The theorem states that if a symmetrical four-port possesses the directional property and is matched at one of its ports, all four ports are matched.

[6] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., New York, N. Y., p. 272; 1945.

$$M = j\begin{bmatrix} Z_B y_{11} & Z_A y_{11}\sqrt{\frac{Z_B}{Z_A}} & -Z_A & 0 \\ -Z_B y_{11}\sqrt{\frac{Z_B}{Z_A}} & -Z_B y_{11} & 0 & Z_B \\ -\frac{1}{Z_A} & 0 & Z_B y_{11} & -Z_B y_{11}\sqrt{\frac{Z_B}{Z_A}} \\ 0 & \frac{1}{Z_B} & Z_A y_{11}\sqrt{\frac{Z_B}{Z_A}} & -Z_B y_{11} \end{bmatrix}. \tag{10}$$

If we are to connect a transmission line of length $\frac{1}{4}\lambda$ to port 1 and one $\frac{3}{4}\lambda$ to port 4 (Fig. 3), the over-all $ABCD$ matrix becomes

$$P = \begin{bmatrix} 1 & 0 & -Z_A Z_B y_{11} & Z_A Z_B y_{11}\sqrt{\frac{Z_B}{Z_A}} \\ 0 & 1 & Z_A Z_B y_{11}\sqrt{\frac{Z_B}{Z_A}} & -Z_B^2 y_{11} \\ -\frac{Z_B}{Z_A} y_{11} & -y_{11}\sqrt{\frac{Z_B}{Z_A}} & 1 & 0 \\ -y_{11}\sqrt{\frac{Z_B}{Z_A}} & -y_{11} & 0 & 1 \end{bmatrix}. \tag{11}$$

which is of the form

$$\begin{bmatrix} I & Dy_{11} \\ Ey_{11} & I \end{bmatrix}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -Z_A Z_B & Z_A Z_B\sqrt{\frac{Z_B}{Z_A}} \\ Z_A Z_B\sqrt{\frac{Z_B}{Z_A}} & -Z_B^2 \end{bmatrix}$$

$$E = -\begin{bmatrix} \frac{Z_B}{Z_A} & \sqrt{\frac{Z_B}{Z_A}} \\ \sqrt{\frac{Z_B}{Z_A}} & 1 \end{bmatrix}.$$

Note that $DE=0$, and $ED=0$.

Let us next cascade $N-1$ such structures, the first characterized by P_1, the second P_2, etc. The over-all matrix becomes

$$P_{N-1}P_{N-2} \cdots P_2P_1 = \begin{bmatrix} I & Dy' \\ Ey' & I \end{bmatrix}$$

where

$$y' = y_{11}^{(1)} + y_{11}^{(2)} + \cdots + y_{11}^{(N-1)}.$$

Finally, to $(N-1)$ such structures we must add, at the end, one more directional filter to complete an N-section cascade directional filter. The over-all network is shown in Fig. 4 and the over-all generalized $ABCD$ matrix is given by

$$MP_{N-1}P_{N-2} \cdots P_1 = M\begin{bmatrix} I & Dy' \\ Ey' & I \end{bmatrix} = j\begin{bmatrix} Z_B Y & Z_A Y\sqrt{\frac{Z_B}{Z_A}} & -Z_A & 0 \\ -Z_B Y\sqrt{\frac{Z_B}{Z_A}} & -Z_B Y & 0 & Z_B \\ -\frac{1}{Z_A} & 0 & Z_B Y & -Z_B Y\sqrt{\frac{Z_B}{Z_A}} \\ 0 & \frac{1}{Z_B} & Z_A Y\sqrt{\frac{Z_B}{Z_A}} & -Z_B Y \end{bmatrix} \tag{12}$$

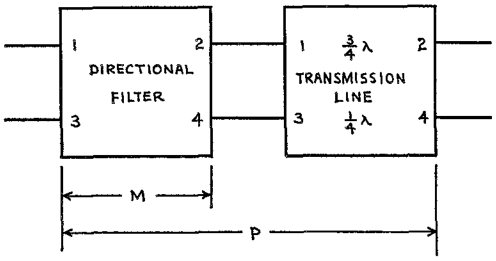

Fig. 3—One unit of a cascade directional filter.

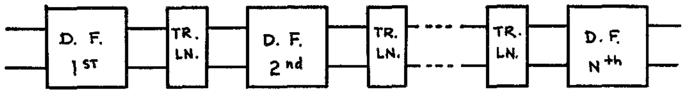

Fig. 4—An N-section cascade directional filter.

where

$$Y = y_{11}^{(1)} + y_{11}^{(2)} + \cdots + y_{11}^{(N)}.$$

Comparing (12) with the $ABCD$ matrix for one directional filter, we see that both are the same except y_{11} in (10) is replaced by Y. Therefore the cascade directional filter is equivalent to one directional filter whose susceptance function y_{11} (or y_{22}) is the sum of the susceptance functions of the component directional filters. The voltage ratios for a cascade directional filter are

$$\frac{V_4}{E} = 0$$

$$\frac{V_1}{E} = \frac{1}{2}$$

$$\frac{V_2}{E} = -\frac{j}{2}\,\frac{1}{1 + Z_B[y_{11}^{(1)} + \cdots + y_{11}^{(N)}]} \tag{13}$$

$$\frac{V_3}{E} = -\frac{1}{2}\sqrt{\frac{Z_B}{Z_A}}\,\frac{Z_B[y_{11}^{(1)} + \cdots + y_{11}^{(N)}]}{1 + Z_B[y_{11}^{(1)} + \cdots + y_{11}^{(N)}]}. \tag{14}$$

Eqs. (13) and (14) suggest that in synthesis, the desired susceptance Y should be expanded in the Foster's partial fraction form, each fraction being realized by one directional filter whose y_{11} is a simple resonant circuit.

In the special case in which all y_{11}'s are identical resonant circuits, the loaded Q of an N-section cascade directional filter is $1/N$ times the loaded Q of one directional filter.

Remarks

This paper has considered only one particular way of cascading the directional filters, namely, they are separated by two sections of transmission line $\frac{1}{4}\lambda$ and $\frac{3}{4}\lambda$ long as shown in Fig. 3. With different lengths, the over-all four-port may have other interesting properties.

The analysis given here is applicable only to directional filters which admit the equivalent circuit representation shown in Fig. 1. As evident from the figure, the unavoidable junction effect in a practical realization has been completely ignored. Moreover, the length of the sections of transmission line may be frequency dependent, as in the case of a hollow guide. In that case, the results contained here are valid only in the "immediate" neighborhood of the average resonant frequency of the various y's. A complete analysis is very complicated even for first-order approximation. We shall leave this problem to an enterprising graduate student.

Appendix I

Derivation of (2)

With reference to Fig. 1, the two series two-ports, which are sections of transmission line $\frac{1}{4}\lambda$ and $\frac{3}{4}\lambda$ long, are characterized by their respective $ABCD$ matrix as follows:

$$\begin{bmatrix} V_{2A} \\ I_{2A} \end{bmatrix} = \begin{bmatrix} A_A & B_A \\ C_A & D_A \end{bmatrix} \begin{bmatrix} V_{1A} \\ I_{1A} \end{bmatrix} = \begin{bmatrix} 0 & -jZ_A \\ -j\dfrac{1}{Z_A} & 0 \end{bmatrix} \begin{bmatrix} V_{1A} \\ I_{1A} \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} V_{2B} \\ I_{2B} \end{bmatrix} = \begin{bmatrix} A_B & B_B \\ C_B & D_B \end{bmatrix} \begin{bmatrix} V_{1B} \\ I_{1B} \end{bmatrix} = \begin{bmatrix} 0 & jZ_B \\ j\dfrac{1}{Z_B} & 0 \end{bmatrix} \begin{bmatrix} V_{1B} \\ I_{1B} \end{bmatrix}. \tag{16}$$

As stated in the text, let the two identical shunt two-ports be characterized by the short-circuit admittance parameters:

$$\begin{bmatrix} I_{1C} \\ I_{2C} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_{1C} \\ V_{2C} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} I_{1D} \\ I_{2D} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_{1D} \\ V_{2D} \end{bmatrix}. \tag{18}$$

The terminal conditions are

$$\begin{aligned} V_{1A} &= V_{2C} = V_1 \\ V_{2A} &= V_{2D} = V_2 \\ V_{1B} &= V_{1C} = V_3 \\ V_{2B} &= V_{1D} = V_4 \\ I_{1A} &= I_1 - I_{2C} \\ I_2 &= I_{2A} - I_{2D} \\ I_{1B} &= I_3 - I_{1C} \\ I_4 &= I_{2B} - I_{1D}. \end{aligned} \tag{19}$$

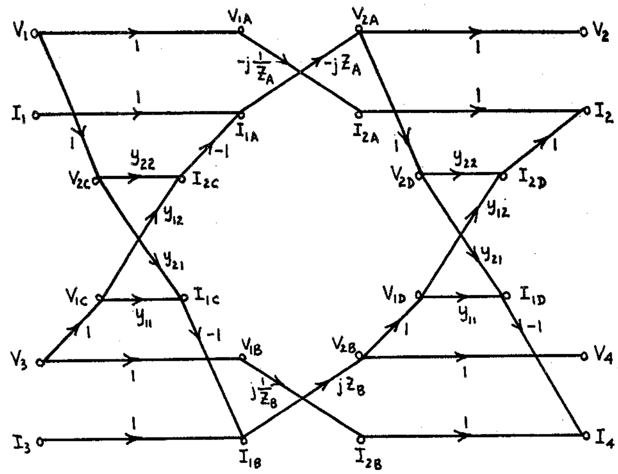

Fig. 5—The signal flow graph of a directional filter (see Fig. 1).

Algebraic manipulation of (15)–(19) to eliminate all variables except V_1, V_2, V_3, V_4, I_1, I_2, I_3 and I_4 leads to (2) in the text. Alternately, one can employ the technique of signal flow graph.[7,8] Fig. 5 shows the signal flow graph which corresponds to (15)–(19). Eq. (2) is obtained by finding the "gains" in the following manner:

$$A_{11} = \text{gain from } V_1 \text{ to } V_2 = jZ_A y_{22}$$

$$A_{12} = \text{gain from } V_3 \text{ to } V_2 = jZ_A y_{12}$$

$$B_{11} = \text{gain from } I_1 \text{ to } V_2 = -jZ_A$$

$$B_{12} = \text{gain from } I_2 \text{ to } V_2 = 0.$$

Appendix II

Derivation of (4)–(9)

The constraints given by (3), which are the terminating conditions on each port, are

$$V_1 = E - I_1 Z_A$$

$$V_2 = Z_A I_2$$

$$V_3 = -Z_B I_3$$

$$V_4 = Z_B I_4.$$

Using these equations and those expressed by (2), one

[7] S. Mason, "Properties of signal flow graph," Proc. IRE, vol. 41, pp. 1144–1156; September, 1953.

[8] S. Mason, "Further properties of signal flow graph," Proc. IRE, vol. 44, pp. 920–926; July, 1956.

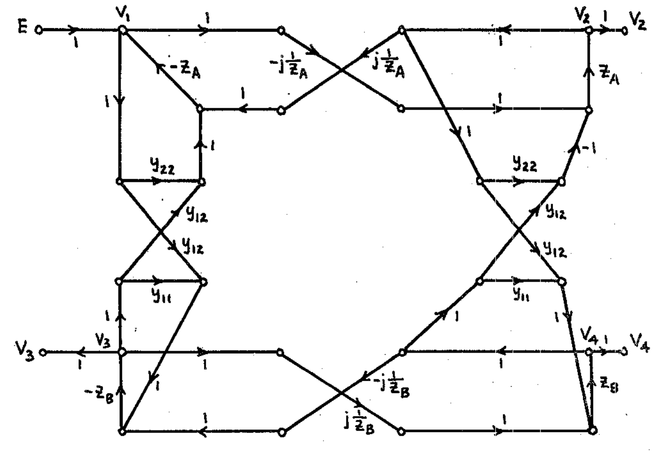

Fig. 6—The signal flow graph of a terminated directional filter.

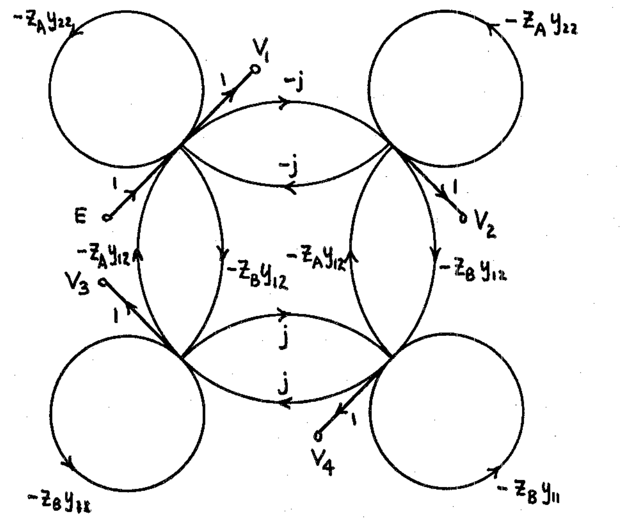

Fig. 7—The simplified version of Fig. 6.

can eliminate all current variables and obtain a set of four equations in four unknowns, V_1, V_2, V_3, V_4. Solving for each in terms of E, we have (4)–(9). The algebra is straightforward though extremely involved. The author is convinced that the signal flow graph technique is superior.

With reference to Fig. 5, we first invert[7] the paths from V_2 to I_1 and from V_4 to I_3. Adding the constraints of (3) to the "inverted" flow graph, we have Fig. 6.

Fig. 6 can be simplified, and the final form is shown in Fig. 7, from which the various "gains" as expressed by (4)–(9) are obtained by inspection.